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Exponential Attractors for a Nonclassical Diffusion Equation with Arbitrary Polynomial Growth*

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Abstract: In this paper, we prove the existence of the exponential attractors for the nonclassical diffusion equations in $H_0^1(\Omega)$ with arbitrary polynomial growth nonlinearity and the distribution derivative term.

Keywords: nonclassical diffusion equations; exponential attractors

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1 Introduction

Let Ω be an open bounded set of $\mathbb{R}^n (n \geq 3)$ with smooth boundary $\partial\Omega$. We consider the following equations

$$\begin{aligned} & u_t - \triangle u_t - \triangle u = f(u) + D_i f^i + g(x), && \text{in} \quad \Omega \times \mathbb{R}_+, \\ & u = 0, && \text{on} \quad \partial\Omega, \\ & u(x,0) = u_0, && x \in \Omega, \end{aligned} \tag{1}$$

where f^i, $g \in L^2(\Omega)\,(i = 1, 2, \cdots, n)$, and D_i is the distribution derivative about the nonlinear functions f. This equation is a special form of the nonclassical diffusion equations used in fluid mechanics, solid mechanics and heat conduction theory[1,2]. Existence of the global attractors for the nonclassical diffusion equations have been studied originally by Kalantarov[3] in $H_0^1(\Omega)$. In recent years, many authors also achieved the existence of the global attractors under different assumptions[4-8]. In [5], the authors obtained the global attractors in $H_0^1(\Omega)$ if the nonlinear function f is a C^1 function and satisfies the following conditions

$$f'(s) \leq l, \quad \forall\, s \in \mathbb{R}, \tag{2}$$

$$-C_1|s|^p - C_0 \leq f(s)s \leq -C_2|s|^p + C_0, \quad p \geq 2, \quad \forall\, s \in \mathbb{R}. \tag{3}$$

We study the existence of exponential attractor for (1) in $H_0^1(\Omega)$ if f satisfies (2) and (3). The main result of this paper is the to-be presented Theorem 3.1.

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2 Preliminaries

We set $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, and the corresponding norms in V and $L^p(\Omega)(1 \le p < \infty)$ are denoted by

$$\|u\|^2 = \int_\Omega |\nabla u|^2, \quad |u|_p^p = \int_\Omega |u|^p.$$

Let X be a separable Hilbert space and $\mathscr{B}$ be a compact subset of X. Let $\{S(t)\}_{t\geq 0}$ be a nonlinear continuous semigroup that leaves the set $\mathscr{B}$ invariant and $\mathscr{A} = \bigcap_{t>0} S(t)\mathscr{B}$, that is, $\mathscr{A}$ is a global attractor for $\{S(t)\}_{t\geq 0}$ on $\mathscr{B}$.

Definition 2.1[9] A compact set $\mathscr{A} \subseteq \mathscr{M} \subseteq \mathscr{B}$ is called an exponential attractor for $(S(t), \mathscr{B})$ if:

1) $\mathscr{M}$ has finite fractal dimension;

2) $\mathscr{M}$ is a positive invariant set of $S(t) : S(t)\mathscr{M} \subseteq \mathscr{M}$, for all $t > 0$;

3) $\mathscr{M}$ is an exponentially attracting set for the semigroup of operators $\{S(t)\}_{t\geq 0}$, i.e., there exist universal constants α, $\beta > 0$ such that, for any $u \in \mathscr{B}$, $\mathrm{dist}_X(S(t)u, \mathscr{M}) \le \alpha e^{-\beta t}$, for all $t > 0$, where dist denotes the nonsymmetric Hausdorff distance between sets.

Definition 2.2[9] A continuous semigroup of operators $\{S(t)\}_{t\geq 0}$ is of the squeezing property on $\mathscr{B}$ if there exists $t_* > 0$ such that $S_* = S(t_*)$ satisfies that there exists an orthogonal projection operator P of rank N_0 such that, for every u and v in $\mathscr{B}$, either

$$\big\|(I-P)\big(S(t_*)u_1 - S(t_*)u_2\big)\big\|_X \le \big\|P\big(S(t_*)u_1 - S(t_*)u_2\big)\big\|_X,$$

or

$$\big\|S(t_*)u_1 - S(t_*)u_2\big\|_X \le \frac{1}{8}\|u_1 - u_2\|_X.$$

Definition 2.3[9] We say $S(t)$ is Lipschitz continuous in the compact set $\mathscr{B}$, if there exists a local bounded function $l(t)$ such that $\|S(t)u - S(t)v\|_X \le l(t)\|u - v\|_X$, for u, $v \in \mathscr{B}$. Here $l(t)$ does not depend on u and v.

3 Exponential attractor in V

Lemma 3.1[5] Let f satisfy the condition (2) and (3), f^i, $g \in L^2(\Omega)\,(i = 1, 2, \cdots, n)$. Then for any $u_0 \in V$ and $T > 0$, the problem (1) has a unique solution u such that $u \in \mathcal{C}([0,T]; V) \bigcap L^\infty(0, \infty; V)$, $u_t \in L^2(0, T; V)$. Moreover, u continuously depends on the initial data in V.

Lemma 3.2[5] There is a positive constant ρ_1 such that, for any bounded subset $B \subset V$, there exists $T = T(B) > 0$, such that $\|S(t)u_0\| + |S(t)u_0|_p \le \rho_1$, for all $t \ge T$ and $u_0 \in B$.

From this lemma, we know that the semigroup of operators $\{S(t)\}_{t\geq 0}$ generalized by (1) possesses a bounded absorbing set $\mathscr{B}_0$ in $H_0^1(\Omega)$. Hence

$$\mathscr{B} = \overline{\bigcup_{0\le t\le T} S(t)\mathscr{B}_0}$$

is a compact invariant positive set in $H_0^1(\Omega)$.

Lemma 3.3[5] There exists positive constants ρ_2 and $T = T(\mathscr{B}_0)$, such that $\|u_t(s)\|^2 + |u_t(s)|_2^2 \le \rho_2$, for all $s \ge T$ and $u_0 \in \mathscr{B}_0$.

We obtain the following results immediately based on Lemma 3.3.

Corollary 3.1 There exists $L > 0$ such that $\sup_{u_0 \in \mathscr{B}} \|u_t(t)\|^2 \le L$, for all $t \ge 0$.

Lemma 3.4[5] Assume that f satisfies (2) and (3), f^i, $g \in L^2(\Omega)\,(i = 1, 2, \cdots, n)$. Then the semigroup $\{S(t)\}_{t \ge 0}$ possesses a global attractor $\mathscr{A}$ in V.

Lemma 3.5 Assume that f satisfies (2) and (3), and $u(t)$, $v(t)$ are two solutions of (1) with initial values u_0, $v_0 \in \mathscr{B}$, respectively, then

$$\|u(t) - v(t)\| \le \mathrm{e}^{c_1 t}\|u(0) - v(0)\|. \tag{4}$$

Proof Setting $w(t) = u(t) - v(t)$, we see that $w(t)$ satisfies

$$w_t - \triangle w_t - \triangle w - \big(f(u) - f(v)\big) = 0. \tag{5}$$

Taking the inner product with w of (5), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(|w|_2^2 + \|w\|^2\big) + \|w\|^2 - \big(f(u) - f(v), w\big) = 0. \tag{6}$$

By (2), it follows that

$$\Big|\int_\Omega (f(u) - f(v))w\mathrm{d}x\Big| \le \int_\Omega \big|f'\big(\theta u + (1-\theta)v\big)\big|\,|w|^2\mathrm{d}x (0 < \theta < 1) \le c_1|w|_2^2.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(|w|_2^2 + \|w\|^2\big) \le c_1\big(|w|_2^2 + \|w\|^2\big).$$

By the Gronwall Lemma, we complete the proof.

Lemma 3.6 Assume the assumptions of Lemma 3.4 hold, then for any $T > 0$, the mapping $(t, u) \mapsto S(t)u$ is Lipschitz continuous on $[0, T] \times \mathscr{B}$.

Proof For u_1, $u_2 \in \mathscr{B}$ and t_1, $t_2 \in [0, T]$ we have

$$\big\|S(t_1)u_1 - S(t_2)u_2\big\| \le \big\|S(t_1)u_1 - S(t_1)u_2\big\| + \big\|S(t_1)u_2 - S(t_2)u_2\big\|. \tag{7}$$

The first term is handled by estimating (4). By virtue of Corollary 3.1, we obtain

$$\big\|u(t_1) - u(t_2)\big\| \le \Big|\int_{t_1}^{t_2} \|u_t(y)\|\mathrm{d}y\Big| \le L_1|t_1 - t_2|. \tag{8}$$

Hence, for some $L = L(T) \ge 0$,

$$\big\|S(t_1)u_1 - S(t_2)u_2\big\| \le L\big[|t_1 - t_2| + \|u_1 - u_2\|\big]. \tag{9}$$

Lemma 3.7 Assume that (2) and (3) hold, and $u(t)$, $v(t)$ are two solutions of (1) with initial values u_0, $v_0 \in \mathscr{B}$, respectively, then the semigroup $S(t)$ satisfies the squeezing property, i.e., there exist t_* and $N = N_0 = N(t_*)$ such that

$$\big\|(I - P)\big(S(t_*)u_0 - S(t_*)v_0\big)\big\| > \big\|P\big(S(t_*)u_0 - S(t_*)v_0\big)\big\|,$$

and

$$\big\|S(t_*)u_0 - S(t_*)v_0\big\| \le \frac{1}{8}\|u_0 - v_0\|.$$

Proof We consider the operator $A = -\Delta$. Since A is self-adjoint, positive operator and has a compact inverse, there exists a complete set of eigenvectors $\{\omega_i\}_{i=1}^{\infty}$ in H, such that the corresponding eigenvalues are $\{\lambda_i\}_{i=1}^{\infty}$, and

$$A\omega_i = \lambda_i\omega_i, \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots \to +\infty, \quad i \to +\infty.$$

We set $\mathrm{H}_N = \mathrm{Span}\{\omega_1, \omega_2, \cdots, \omega_N\}$. P_N is the orthogonal projection onto H_N, and $Q_N = I - P_N$ is the orthogonal projection onto the orthogonal complement of H_N, and $w = P_N w + Q_N w \triangleq p + q$. Assume that $\|P_N w(t)\| \leq \|Q_N w(t)\|$, taking the inner product of (5) with q, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|q|_2^2 + \|q\|^2) + \|q\|^2 - (f(u) - f(v), q) = 0. \tag{10}$$

By (2), it leads to

$$\left|\int_\Omega (f(u) - f(v))q\mathrm{d}x\right| \leq c_2 \int_\Omega |w|\,|q|\mathrm{d}x \leq \frac{\|q\|^2}{2} + \frac{c_2}{2}|w|_2^2. \tag{11}$$

Combining (10) with (11), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}(|q|_2^2 + \|q\|^2) + \|q\|^2 \leq c_2|w|_2^2. \tag{12}$$

Furthermore, by Lemma 3.5 and the Poincaré inequality, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(|q|_2^2 + \|q\|^2) + \frac{\|q\|^2}{2} + \frac{\lambda_{N+1}}{2}|q|_2^2 \\ \leq\ & c_2|w|_2^2 \leq c_2|p+q|_2^2 \leq 2c_2|q|_2^2 \leq 2c_2\lambda_{N+1}^{-1}\|q\|^2 \\ \leq\ & c_3\lambda_{N+1}^{-1}\|w\|^2 \leq c_3\lambda_{N+1}^{-1}\mathrm{e}^{c_1 t}\|w(0)\|^2. \end{aligned} \tag{13}$$

Since $\lambda_1 \leq \lambda_{N+1}$, let $c_4 = \min\{\frac{1}{2}, \frac{\lambda_1}{2}\}$, and then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(|q|_2^2 + \|q\|^2) + c_4(|q|_2^2 + \|q\|^2) \leq c_3\lambda_{N+1}^{-1}\mathrm{e}^{c_1 t}\|w(0)\|^2. \tag{14}$$

By the Gronwall Lemma, we conclude that

$$\begin{aligned} |q(t)|_2^2 + \|q(t)\|^2 &\leq \mathrm{e}^{-c_4 t}(|q(0)|_2^2 + \|q(0)\|^2) + c_5\lambda_{N+1}^{-1}\mathrm{e}^{c_1 t}\|w(0)\|^2 \\ &\leq c_6(\mathrm{e}^{-c_4 t} + c_7\lambda_{N+1}^{-1}\mathrm{e}^{c_1 t})\|w(0)\|^2. \end{aligned}$$

Hence

$$\|w(t)\|^2 \leq 2\|q(t)\|^2 \leq c_8(\mathrm{e}^{-c_4 t} + c_9\lambda_{N+1}^{-1}\mathrm{e}^{c_1 t})\|w(0)\|^2.$$

Let $t_* > 0$, such that $c_8 e^{-c_4 t_*} \leq \frac{1}{128}$, and then let t_* be fixed, and N large enough, such that

$$c_8 c_9 \lambda_{N+1}^{-1}\mathrm{e}^{c_1 t_*} \leq \frac{1}{128}.$$

Thus, we obtain that

$$\|w(t_*)\| \leq \frac{1}{8}\|w(0)\|.$$

Theorem 3.1 Assume that $f \in C^1(\mathbb{R}^1;\mathbb{R}^1)$ satisfies (2) and (3), $f^i, g \in L^2(\Omega)\,(i = 1,2,\cdots,n)$. Then there exists an exponential attractor $\mathscr{M} \subset V$ for the semigroup $\{S(t)\}_{t\geq 0}$ generated by (1).

Proof From Lemma 3.7, $S(t_*)$ satisfies the squeezing property for some $t_* > 0$. According to Theorem 2.1 in [9], there exists an exponential attractor $\mathscr{M}_*$ for $(S(t_*),\mathscr{B})$ and we set

$$\mathscr{M} = \bigcup_{0\leq t\leq t_*} S(t)\mathscr{M}_*.$$

By Lemma 3.6, $(t,u) \mapsto S(t)u$ is Lipschitz continuous from $[0,T] \times \mathscr{B}$ to $\mathscr{B}$, and by the proof of Theorem 3.1 in [9], it is easy to see that $\mathscr{M}$ is an exponential attractor for $(\{S(t)\}_{t\geq 0},\mathscr{B})$.

References:

[1] Aifantis E C. On the problem of diffusion in solids[J]. Acta Mech, 1980, 37: 265-296

[2] Lions J L, Magenes E. Non-homogeneous Boundary Value Problems and Applications[M]. Berlin: Spring-Verlag, 1972

[3] Kalantarov V K. On the attractors for some non-linear problems of mathematical physics[J]. Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta, 1986, 152: 50-54

[4] Xiao Y L. Attractors for a nonclassical diffusion equation[J]. Acta Math Appl Sinica, 2002, 18: 273-276

[5] Sun C Y, Wang S Y, Zhong C K. Global attractors for a nonclassical diffusion equation[J]. Acta Math Sinica English series, 2007, 23: 1271-1280

[6] Ma Q Z, Zhong C K. Global attractors of strong solutions to nonclassical diffusion equations[J]. Journal of Lanzhou University, 2004, 40: 7-9

[7] Wang S Y, Li D S, Zhong C K. On the dynamics of a class of nonclassical parabolic equations[J]. Journal of Mathematical Analysis and Applications, 2006, 317: 565-582

[8] Temam R. Infinite-dimensional Dynamical Systems in Mechanics and Physics[M]. New York: Springer, 1997

[9] Eden A, Foias C, Nicolaenko B, *et al.* Exponential Attractors for Dissipative Evolution Equations[M]. New York: Masson Paris Wiely, 1994

非线性项任意次多项式增长时非经典反应扩散方程的指数吸引子

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摘 要: 本文证明了带导数项的非经典反应扩散方程当非线性项任意多项式增长时在 $H_0^1(\Omega)$ 中指数吸引子的存在性.

关键词: 非经典反应扩散方程；指数吸引子